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LETTER TO THE EDITOR

On the $1/N$ corrections to the Green functions of random matrices with independent entries

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Abstract. We propose a general approach to the construction of $1/N$ corrections to the Green function $G_N(z)$ of the ensembles of random real-symmetric and Hermitian $N \times N$ matrices with independent entries $H_{k,l}$. By this approach we study the correlation function $C_N(z_1, z_2)$ of the normalized trace $N^{-1}\mathrm{Tr}\ G_N$ assuming that the average of $|H_{k,l}|^5$ is bounded. We found that to the leading order $C_N(z_1, z_2) = N^{-2}F(z_1, z_2)$, where $F(z_1, z_2)$ only depends on the second and the fourth moments of $H_{k,l}$. For the correlation function of the density of energy levels we obtain an expression which, in the scaling limit only depends on the second moment of $H_{k,l}$. This can be viewed as supporting the universality conjecture of random matrix theory.

Random $N \times N$ matrices with independent entries were introduced by Wigner [1]. The majority of the rigorous results for these matrices concern the proof of convergence of the density $\rho_N(E)$ of their eigenvalues to the celebrated Wigner semicircle law and its generalization, known as the deformed semicircle law [2–4]. These non-random limiting eigenvalue distributions are completely determined by the two first moments of the probability distribution of random matrix entries.

Much less well known are the large-N corrections, in particular, their dependence on the probability distribution of the entries. The aim of this letter is to present a rigorous approach to the systematic construction of the large-N corrections for moments of the Green functions of respective random matrices.

We consider an ensemble of real-symmetric $N \times N$ random matrices which, in the particular case of the Gaussian distributed entries, possessing the orthogonal invariance property (the respective ensemble is known as the Gaussian orthogonal ensemble (GOE) and is an archetype model in the field [5]). Thus, our random matrices $H = \{H_{k,l}\}_{k,l=1}^{N}$ are specified by relations

$$H_{k,l} = H_{l,k} = \left(\frac{1+\delta_{k,l}}{N}\right)^{1/2} W_{k,l} \qquad k, l = 1, \ldots, N. \tag{1}$$

Here $W_{k,l}$, $k \leqslant l$, are independent random variables such that: (i) $\overline{W_{k,l}} = 0$; (ii) $\overline{W_{k,l}^2} \equiv v^2$; (iii) $\overline{W_{k,l}^4} - 3(\overline{W_{k,l}^2})^2 \equiv \sigma$; and (iv) $\max_{k \leqslant l} \overline{|W_{k,l}|^5} \leqslant C$, where the bar denotes averaging over the respective probability distribution and v, σ and C are N-independent. Thus $W_{k,l}$'s may have different probability distributions for different pairs (k, l), but their second and fourth moments have to be the same.

We denote by $G_N(z) = \{G_{i,k}(z)\}_{i,k=1}^N$ the Green function $G_N(z) = (H_N - z)^{-1}$, $\mathrm{Im}\, z \neq 0$, of our matrices and by $\langle G_N(z)\rangle$ its normalized trace $\langle G_N(z)\rangle = 1/N \sum_{k=1}^N G_{k,k}(z)$. It is known [4, 6] that the moments of $\langle G_N(z)\rangle$ factorize. Namely, if

$$m_N^{(p)}(z_1, \dots, z_p) = \overline{\prod_{i=1}^N \langle G_N(z_i)\rangle} \tag{2}$$

then

$$m_N^{(p)}(z_1, \dots, z_p) = \prod_{i=1}^p m_N^{(1)}(z_i) + \mathrm{O}(N^{-2}). \tag{3}$$

In particular, the correlation function

$$C_N(z_1, z_2) = \overline{\langle G_N(z_1)\rangle\langle G_N(z_2)\rangle} - \overline{\langle G_N(z_1)\rangle}\;\overline{\langle G_N(z_2)\rangle} \tag{4}$$

for $\mathrm{Im}\, z_1, \mathrm{Im}\, z_2 \neq 0$ is of the order of $1/N^2$ as $N \to \infty$.

Our aim is to construct the expansion of $m_N^{(p)}(z_1, \dots, z_p)$ in powers of $1/N$. We outline the main idea and find, as an example, the explicit form of the $1/N^2$ correction to $C_N(z_1, z_2)$. We also discuss some implications of our results.

Our approach is an adaptation and extension of that proposed in [7] for studying spectral characteristics of a certain class of random finite-difference operators of order R, acting in $l^2(\mathbf{Z}^d) \otimes \mathbf{C}^n$ in the limiting cases when one of the parameters R, n, or d tends to infinity. The main idea is to derive certain identities for the moments $m_N^{(p)}(z_1, \dots, z_p)$ and then, treating the whole set of these relations as an equation in an appropriate linear space, to compute the moments in each order of $1/N$ by iterating this equation.

To derive the relations let us rewrite (2) as $1/N \sum_{k=1}^N \overline{G_{k,k}(z_1) \prod_{i=2}^p \langle G_N(z_i)\rangle}$ and replace $G_{k,k}(z_1)$ by the RHS of the resolvent identity $G_{k,k}(z_1) = -1/z_1 + 1/z_1 \sum_{l=1}^N H_{k,l} G_{l,k}(z_1)$. We obtain

$$m_N^{(p)}(z_1, \dots z_p) = -\frac{1}{z_1} m^{(p-1)}(z_2, \dots z_p) + \sum_{k,l=1}^N \overline{H_{k,l} G_{l,k}(z_1) \prod_{i=2}^p \langle G_N(z_i)\rangle} \tag{5}$$

where for $p = 0$ we set $m_N^{(0)} \equiv 1$. Now we average in the second term of the RHS of (5) over the random variable $H_{k,l}$ by using another resolvent identity

$$G_{m,n} = \hat{G}_{m,n} - \left[\hat{G}_{m,k} G_{l,n} - \hat{G}_{m,l} G_{k,n}\right] H_{k,l} (1 + \delta_{k,l})/2 \tag{6}$$

where $\hat{G} = G\,\big|_{H_{k,l}=0}$. Iterating (5) several times we represent any matrix element of $G_N(z)$ as a sum of powers of $H_{k,l}$ multiplied by matrix elements of $\hat{G}_N$ and of a power of $H_{k,l}$, say $H_{k,l}^a$, multiplied by a sum of matrix elements of both Green functions G_N and $\hat{G}_N$ (a is determined by the number of iterations). The inequality

$$|G_{k,m}(z)| \leqslant \|G_N\| \leqslant |\mathrm{Im}\, z|^{-1} \tag{7}$$

and analogous inequality for $\hat{G}_N$ allows us to estimate the average of the latter by $\overline{|H_{k,l}|^a}$ (which is proportional to $1/N^{a/2}$ according to (1)) multiplied by a power of $|\mathrm{Im}\, z|^{-1}$ and by some absolute constant. Since $\hat{G}_N$ and $H_{k,l}$ are independent, all other terms in this representation will have the form of moments of matrix elements of $\hat{G}_N$ multiplied by moments of $H_{k,l}$ whose order is smaller than a. At last, we return back from $\hat{G}_N$ to G_N by using the identity that differs from (6) by interchanging G_N and $\hat{G}_N$ and replacing $H_{k,l}$ by $-H_{k,l}$. We obtain the identity

$$m_N^{(p)} = (A m_N)^{(p)} + f_N^{(p)}. \tag{8}$$

Here A is the linear operator defined as

$$(Am)^{(p)}(z_1,\ldots,z_p) = -\frac{1-\delta_{1,p}}{z_1}m^{(p-1)}(z_2,\ldots,z_p) - \frac{v^2}{z_1}m^{(p+1)}(z_1,z_1,\ldots,z_p)$$

in the Banach space B of bounded sequences $\{m^{(p)}\}_{p=1}^{\infty}$, $\|m\| = \sup_{p\geqslant 1} v^p|m^{(p)}|$. $f_N^{(p)}$ for every p is a sum of moments whose form is different from (2).

If $|\mathrm{Im}\ z_i| > 2v$, then $\|A\| \leqslant 2v/|\mathrm{Im}\ z_i| < 1$, the operator $1 - A$ is invertible and (8) has a unique solution. Since according to (7) the moments (2) are bounded above by $\prod_{i=1}^{p}|\mathrm{Im}\ z_i|^{-1}$, this solution coincides with $\{m^{(p)}(z_1,\ldots,z_p)\}_{p=1}^{\infty}$ under the above condition $|\mathrm{Im}\, z_i| > 2v$.

Furthermore, to the leading order $f_N^{(p)} = -\delta_{p,1}/z_1$ and therefore (cf (3))

$$m_N^{(p)}(z_1,\ldots,z_p) = \prod_{i=1}^{p} r(z_i) + \mathrm{O}(N^{-1}) \tag{9}$$

where $r(z) = (-z + \sqrt{z^2-4v^2})/2v^2$ is the Stieltjes transform of the semicircle law, i.e. $\pi^{-1}\mathrm{Im}\ r(E+\mathrm{i}0) = {}_{+}\sqrt{4v^2-E^2}/2\pi v^2$; where ${}_{+}\sqrt{t} = \sqrt{\max(t,0)}$.

Next orders of $f_N^{(p)}$ in the Gaussian case are given by expressions

$$\frac{v^2}{z_1 N}\overline{\left\langle G_N(z_1)^2\right\rangle \prod_{j=2}^{p}\left\langle G_N(z_j)\right\rangle} \qquad \frac{2v^2}{z_1 N^2}\sum_{j=2}^{p}\overline{\left\langle G_N(z_1)G_N(z_j)^2\right\rangle \prod_{k=2,k\neq j}^{p}\left\langle G_N(z_k)\right\rangle}$$

etc. We see that moments of the form different from (2) do appear in these expressions. These new moments of can be found by an argument analogous to that for $m_N^{(p)}$, i.e. by deriving equations similar to (8). In the general case there are additional terms in each order of $1/N$. They can be handled analogously (see, for example, the first term in the RHS of (10) below). Therefore, solving (8) and these 'higher order' equations step by step in each order of $1/N$ we obtain corrections to $m_N^{(p)}$ for any p.

For the Gaussian entries $1/N$ corrections were studied in the physical paper [8] based on the formal perturbation theory with respect to $H_{k,l}$ and the diagrammatic technique. This approach is an adaptation of respective technique developed in [9] in order to construct the $1/N$-expansion of the Green function moments of the random operator describing a disordered system on Z^d with N orbitals at each site. It is not an easy problem to extend this technique (especially in its rigorous version) to the non-Gaussian case. In comparison with this our approach is much less sensitive to the type of probability distribution of $H_{k,l}$. Moreover, the complicated and cumbersome combinatorial problem of rearranging of diagrams does not appear. In particular, the 'dressing' procedure replacing the 'bare' Green function $-1/z$ by $\lim_{N\to\infty}\langle G_N(z)\rangle$ is automatic in our approach.

This is especially evident in the computation of the two-point correlation function (4). Here we can simplify our general procedure because in this case it is sufficient to iterate only the few first relations of the infinite system (8). Namely, if as before, $|\mathrm{Im}\ z_i| > 2v$, then

$$\begin{aligned}\left(z_1 - 2v^2\overline{\langle G_N(z_1)\rangle}\right)C_N(z_1,z_2) &= \frac{2\sigma}{N^2}\sum_{k,l=1}^{N}\overline{G_{k,k}(z_1)G_{k,k}(z_2)G_{l,l}(z_1)G_{l,l}(z_2)^2}\\ &\quad + \frac{2v^2}{N^2}\overline{\langle G_N(z_1)G_N(z_2)^2\rangle} + \mathrm{O}\left(N^{-5/2}\right).\end{aligned} \tag{10}$$

Due to the factor $1/N^2$ in front of the first and the second terms of the RHS of this relation we can replace respective averages in these terms by their limiting (zero-order) values given by the first term in the RHS of (8). We obtain after some algebra

$$C_N(z_1, z_2) = N^{-2}F(z_1, z_2) + O(N^{-5/2}) \tag{11}$$

$$F(z_1, z_2) = \frac{1}{N^2}\left(\frac{2v^2(r_1 - r_2)^2}{\beta(z_1 - z_2)^2(1 - v^2r_1^2)(1 - v^2r_2^2)} + \frac{2\sigma r_1^3 r_2^3}{(1 - v^2r_1^2)(1 - v^2r_2^2)}\right) + O(N^{-5/2}) \tag{12}$$

where $\beta = 1$.

Notice that by an analogous argument we can also find $1/N$ corrections to the $N = \infty$ limit of the Green functions of an ensemble of the Hermitian random matrices. For instance, let

$$H_{k,l} = N^{-1/2}(X_{k,l} + \mathrm{i}Y_{k,l}) \qquad k, l = 1 \ldots N \tag{13}$$

where $X_{k,l} = X_{l,k}$ and $Y_{k,l} = -Y_{l,k}$ are independent for $k \leqslant l$ random variables such that (i) $\overline{X_{k,l}} = \overline{Y_{k,l}} = 0$; (ii) $\overline{X_{k,l}^2} = (1 + \delta_{k,l})v^2/2$ and $\overline{Y_{k,l}^2} = (1 - \delta_{k,l})v^2/2$; (iii) $\overline{X_{k,l}^4} - 3(\overline{X_{k,l}^2})^2 = \sigma_X$, $\overline{Y_{k,l}^4} - 3(\overline{Y_{k,l}^2})^2 = \sigma_Y$, $\sigma_X + \sigma_Y = \sigma$, if $k < l$, and $\overline{X_{k,k}^4} - 3(\overline{X_{k,k}^2})^2 = \sigma$ and (iv) $\sup_{k \leqslant l} \overline{|X_{k,l}|^5 + |Y_{k,l}|^5} \leqslant C < \infty$, where v, σ, and C are N-independent. Then the two-point correlation function of $1/N \operatorname{Tr} G_N(z)$ is given by (11) and (12) with $\beta = 2$.

Above we have presented the scheme of rigorous construction of $1/N$ corrections (in fact expansions) for moments of normalized traces of the Green functions of random matrix ensembles (1) and (13). Now we use our result to draw certain non-rigorous conclusions on the form of the leading term of the correlation function $S_N(E_1, E_2)$ of the density of states (DOS) $\rho_N(E) = N^{-1} \operatorname{Tr} \delta(H - E) \equiv \langle\delta(H - E)\rangle$. Since $\rho_N(E) = 1/\pi \lim_{\epsilon \downarrow 0} \operatorname{Im}\langle G_N(E + \mathrm{i}\epsilon)\rangle \equiv I_E\{G_N(z)\}$, we see that to obtain $S_N(E_1, E_2)$ we have to use (11) and (12) outside the domain $|\operatorname{Im} z| > 2v$ where they were rigorously proved. Nevertheless, since the function $F(z_1, z_2)$ given by (12) can obviously be continued up to the real axis with respect to both the variables z_1 and z_2 we can apply the operations I_{E_1} and I_{E_2}, $E_1 \neq E_2$ to this expression. It means that we are going to compute the leading term of the DOS correlation function by first performing the limit $N \to \infty$ and then the limits $\epsilon_1, \epsilon_2 \downarrow 0$. This order of limiting transitions is the inverse with respect to that prescribed by the definition of this correlation function.

The resulting expression for the correlation function is

$$S_N(E_1, E_2) = -\frac{1}{\beta\pi^2[N(E_1 - E_2)]^2} \frac{4v^2 - E_1E_2}{\sqrt{4v^2 - E_1^2}\sqrt{4v^2 - E_2^2}} - \frac{2\sigma}{N^2\pi^2v^6} \frac{(2v^2 - E_1^2)(2v^2 - E_2^2)}{\sqrt{4v^2 - E_1^2}\sqrt{4v^2 - E_2^2}}. \tag{14}$$

For the Gaussian orthogonal and unitary ensembles (GOE and GUE) $\sigma = 0$ and we recover the result

$$S_N(E_1, E_2) = -\frac{1}{\beta\pi^2[N(E_1 - E_2)]^2} \frac{4v^2 - E_1E_2}{\sqrt{4v^2 - E_1^2}\sqrt{4v^2 - E_2^2}} \tag{15}$$

obtained in [10, 11]. We see that in a general non-Gaussian case the respective expression depends not only on the second moment of random entries but also on the fourth moment via σ.

Now let us consider the so-called scaling limit of $S_N(E_1, E_2)$, when $E_1, E_2 \to E, N(E_1 - E_2) \to s$ [5]. We obtain a remarkably simple expression: $\lim_{N(E_1-E_2)\to s} S_N(E_1, E_2) = -1/(\beta\pi^2 s^2)$. According to Wigner and Dyson (see, for example, [5]) the exact large-s asymptote of the DOS correlation functions of the Gaussian ensembles are: $-1/(\pi^2 s^2)$ (GOE) and $-\sin^2 \pi\rho(E)s/(\pi^2 s^2)$ (GUE). Comparing these expressions with ours we see that our procedure of computing of the correlation function yields, for the general case, an expression coinciding with the large-s asymptote of the Gaussian correlation function smoothed over the energy intervals $\Delta s \gg \rho^{-1}(E)$. This can be regarded as support of the universality conjecture of random matrix theory [5].

Let us mention three more supports of this conjecture. The first one [12] concerns the so-called sparse (or diluted) random matrices whose entries are independently distributed random variables such that $\Pr\{H_{k,l} = 0\} = p/N$. The authors used the Grassman integral technique and found the Wigner–Dyson universal form of the DOS correlator if p is large enough. The second was obtained recently [13] for the completely different ensemble, known as the unitary invariant ensemble whose probability density is $Z^{-1} \exp[-NV(H)]$, where $V(t)$ is an even polynomial. Based on an approach known as the orthogonal polynomial technique, the authors established a number of interesting results concerning the eigenvalue statistics of this ensemble, in particular, the relation (15) for $\beta = 2$. The third was obtained by the present authors for the ensemble $H = \sum_{\mu=1}^{p} \tau_\mu(\cdot, \xi^\mu)\xi^\mu$, where τ_μ and $\xi^\mu = \{\xi_1^\mu, \ldots, \xi_n^\mu\}$ are independent identically-distributed random variables. For this ensemble which was introduced in [14] we obtained the analogue of (11) and (12) and showed that its scaling limit is the same as above. These results will be published elsewhere.

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References

[1] Wigner E 1958 *Ann. Math.* **57** 168
[2] Pastur L 1973 *Russ. Math. Surveys* **28** 1
[3] Brody T A, Flores J, French J B, Mello P A, Pandey A and Wong S S M 1981 *Rev. Mod. Phys.* **53** 385
[4] Girko V L 1988 *Spectral Theory of Random Matrices* (Moscow: Nauka) (in Russian)
[5] Mehta M L 1991 *Random Matrices* 2nd edn (New York: Academic)
[6] Khorunzhy A M, Khoruzhenko B A, Pastur L A and Shcherbina M V 1992 *Phase Transitions and Critical Phenomena* vol 15 ed C Domb and J L Lebowitz (London: Academic) p 73
[7] Khorunzhy A M and Pastur L A 1993 *Commun. Math. Phys.* **153** 605
[8] Verbaarschot J J M, Weidenmuller H A and Zirnbauer M R 1984 *Ann. Phys., NY* **153** 367
[9] Opermann R and Wegner F 1979 *Z. Phys.* B **34** 327
[10] French J B, Mello P A and Pandey A 1978 *Ann. Phys., NY* **113** 277
[11] Pandey A 1981 *Ann. Phys., NY* **134** 110
[12] Mirlin A D and Fyodorov Ya N 1991 *J. Phys. A: Math. Gen.* **24** 227
[13] Brezin E and Zee A 1993 *Nucl. Phys.* B **402** 613
[14] Marchenko V and Pastur L 1967 *Mat. Sbornik* **72** 507